Nonlinear lattices and gap solitons

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(Received 24 February 1993)

Strong similarities between the properties of nonlinear diatomic lattices, recently explored by Kivshar and Flytzanis [Phys. Rev. A **46**, 7972 (1992)], and those of periodic nonlinear optical structures are elucidated.

PACS number(s): 03.40.Kf, 63.50.+x, 66.90.+r, 42.25.−p

In a recent article Kivshar and Flytzanis analyze the properties of nonlinear diatomic lattices [1]. They show that such lattices support solitonlike solutions, which are somewhat similar to the gap-soliton solutions which have been investigated in periodic nonlinear optical structures. Here I would like to point out that, in fact, the similarity between the nonlinear diatomic lattices and periodic nonlinear optical structures is actually very strong, satisfying very similar equations. Specifically, the authors consider a nonlinear lattice with period a, in which the displacement $u_n(t)$ of atom n satisfies the equation of motion

$$m_n \frac{d^2 u_n}{dt^2} + K(2u_n - u_{n+1} - u_{n-1}) + \alpha u_n + \beta u_n^3 = 0, \quad (1)$$

where m_n is the mass of atom n, K is the coupling constant, and α and β are linear and nonlinear parameters, respectively. Taking $m_n = m$ for n even, while $m_n = M$ $(M > m)$ for n odd, leads, in the linear limit, to the well known dispersion relation for a diatomic lattice exhibiting a low-frequency acoustic branch, and higher-frequency optical branch [2]. The authors now concentrate on the properties of such a system at the edge of the Brillouin zone where the wave number $q = \pi/2a$, and where, in the linear limit, the two branches have the associated frequencies $\omega_{1,2}$, where

$$\omega_1^2 = \frac{\alpha + 2K}{M}, \qquad \omega_2^2 = \frac{\alpha + 2K}{m}. \quad (2)$$

The particles' motion is such that at ω_1 (ω_2) the light (heavy) particles are at rest, while the heavy (light) particles oscillate with opposite phase [1,2].

To study the properties of the *nonlinear lattice* the authors make the common assumption that the particles' motion is only slightly modified by the nonlinearity. This modification is expressed by two envelope functions, one for the sublattice consisting of the light particles, and one for that consisting of the heavy particles, $v_n(t)$ and $w_n(t)$, respectively. For small nonlinearities these envelopes are slowly varying and can be considered to be continuous functions of position, and thus $v_n(t) = v(x,t)$, and similarly for w. The authors then show that these envelope functions satisfy [1]

$$i m \omega_1 \frac{\partial v}{\partial t} + \frac{1}{2} m \Delta\omega^2 v - aK \frac{\partial w}{\partial x} - \frac{3}{2}\beta |v|^2 v = 0,$$
$$i M \omega_1 \frac{\partial w}{\partial t} + aK \frac{\partial v}{\partial x} - \frac{3}{2}\beta |w|^2 w = 0. \quad (3)$$

The authors then proceed to find various solitonlike solutions to these equations.

Although Kivshar and Flytzanis point out the similarity between their work and the investigation of periodic nonlinear optical media, this is not all clear from Eqs. (3). Here I would like to stress that this relation is very strong, and that Eqs. (3) can be transformed such that the similarity is evident. To do so it is easiest, although not essential, to consider the limit in which the mass difference $M - m$ is small, so that $M \approx m \equiv \mu$, and $\omega_1 \approx \omega_2 \approx \bar{\omega} \equiv \frac{1}{2}(\omega_1 + \omega_2)$. We therefore may set $\Delta\omega^2 = (\omega_2 + \omega_1)(\omega_2 - \omega_1) = 2\bar{\omega}\delta$, where δ is the width of the forbidden zone, so that we find

$$i\mu\bar{\omega} \frac{\partial v}{\partial t} + \mu\bar{\omega}\delta v - aK \frac{\partial w}{\partial x} - \frac{3}{2}\beta |v|^2 v = 0,$$
$$i\mu\bar{\omega} \frac{\partial w}{\partial t} + aK \frac{\partial v}{\partial x} - \frac{3}{2}\beta |w|^2 w = 0. \quad (4)$$

I now introduce the following definitions:

$$V = \frac{aK}{\mu\bar{\omega}}, \qquad \kappa = \frac{\delta}{2V}, \qquad \Gamma = -\frac{3}{8}\frac{\beta}{\mu\omega V}. \quad (5)$$

It is easy to demonstrate that V equals the group velocity at the Brillouin zone edge in the limit we are considering. Defining further the new functions $f_\pm$ through

$$f_\pm = (v \pm i w) e^{\frac{i}{2}\delta t}, \quad (6)$$

it is straightforward to show that these new functions satisfy the set of coupled equations

$$+i \frac{\partial f_+}{\partial x} + \frac{i}{V}\frac{\partial f_+}{\partial t} + \kappa f_- + \Gamma(|f_+|^2 f_+ + 2|f_-|^2 f_+ + f_-^2 f_+^*) = 0,$$

$$-i \frac{\partial f_-}{\partial x} + \frac{i}{V}\frac{\partial f_-}{\partial t} + \kappa f_+ + \Gamma(|f_-|^2 f_- + 2|f_+|^2 f_- + f_+^2 f_-^*) = 0. \quad (7)$$

Now these equations are very similar to the coupled mode equations which describe the properties of periodic nonlinear optical media in the limit in which the periodicity is weak [3,4]. There the $f_\pm$ represent the slowly varying envelopes of the amplitudes of the forward and backward propagating modes. In fact the only differ-

1063-651X/93/48(5)/4136(2)/$06.00

ence between Eqs. (7) and those for periodic nonlinear optical media is the presence of the last term in each of Eqs. (7); such "phase conjugation" terms are well known in nonlinear optics. By a phase matching argument applied to the $f_{\pm}$ these phase conjugation terms can be shown to be unimportant in the optical context. In nonlinear lattices such terms evidently survive—this is so because here phase matching arguments are applied to the envelope function v and w [1], not to the $f_{\pm}$. As a consequence, different terms "survive," thus ultimately resulting in slightly different equations of motion. In the basis of the new envelope functions $f_{\pm}$, the mode coupling arises from gratinglike terms, which are proportional to κ, as well as from nonlinear cross-phase modulation and phase-conjugation terms. The relation between the properties of nonlinear diatomic lattices and those of periodic nonlinear optical structures can be further elucidated if one realizes that the envelope functions $v(x,t)$ and $w(x,t)$ in fact modulate the Bloch functions of the linear periodic lattice.

By a simple transformation I have thus explicitly shown that there is a very close relation between the properties of periodic nonlinear optical media and those of nonlinear diatomic lattices. I intend to explore these similarities in more depth in future publications.

The author gratefully acknowledges the hospitality of Professor J. E. Sipe from the University of Toronto, where most of this work was done. The Optical Fibre Technology Centre is part of the Australian Photonics CRC.

[1] Y.S. Kivshar and N. Flytzanis, Phys. Rev. A **46**, 7972 (1992).

[2] See, e.g., C. Kittel, *Introduction to Solid State Physics*, 5th ed. (Wiley, New York, 1976), Chap. 4.

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[4] C.M. de Sterke and J.E. Sipe, Phys. Rev. A **42**, 550 (1990).